

Polynomial Representation of F_4 and a New Combinatorial Identity about Twenty-Four¹

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Abstract

Singular vectors of a representation of a finite-dimensional simple Lie algebra are weight vectors in the underlying module that are nullified by positive root vectors. In this article, we use partial differential equations to find all the singular vectors of the polynomial representation of the simple Lie algebra of type F_4 over its basic irreducible module. As applications, we obtain a new combinatorial identity about the number 24 and explicit generators of invariants. Moreover, we show that the number of irreducible submodules contained in the space of homogeneous harmonic polynomials with degree $k \geq 2$ is $\geq \lfloor k/3 \rfloor + \lfloor (k-2)/3 \rfloor + 2$.

1 Introduction

It has been known for many years that the representation theory of Lie algebra is closely related to combinatorial identities. Macdonald [M] generalized the Weyl denominator identities for finite root systems to those for infinite affine root systems, which are now known as the Macdonald's identities. Lepowsky and Garland [LG] gave a homologic proof of Macdonald's identities. Kac (e.g., cf [Ka]) derived these identities from his generalization of Weyl's character formula for the integrable representations of affine Kac-Moody algebras, known as Weyl-Kac formula. Lepowsky and Wilson [LW1, LW2] found a representation theoretic proof of the Rogers-Ramanujan identities. There are a number of the other works relating combinatorial identities to representations of Lie algebras.

¹2000 Mathematical Subject Classification. Primary 17B10, 17B25; Secondary 17B01.

²Research supported by China NSF 10871193

We present here a consequence of the Macdonald's identities taken from Kostant's work [Ko1]. Let \mathcal{G} be a finite-dimensional simple Lie algebra over the field \mathbb{C} of complex numbers. Denote by Λ^+ the set of dominant weights of \mathcal{G} and by $V(\lambda)$ the finite-dimensional irreducible \mathcal{G} -module with highest weight λ . It is known that the Casimir operator takes a constant $c(\lambda)$ on $V(\lambda)$. Macdonald's Theorem implies that there exists a map $\chi : \Lambda^+ \rightarrow \{-1, 0, 1\}$ such that the following identity holds:

$$\left(\prod_{n=1}^{\infty} (1 - q^n)\right)^{\dim \mathcal{G}} = \sum_{\lambda \in \Lambda^+} \chi(\lambda) (\dim V(\lambda)) q^{c(\lambda)}. \quad (1.1)$$

Kostant [K2] found a connection of the above identity with the abelian subalgebras of \mathcal{G} .

The number 24 is important in our life; for instance, we have 24 hours a day. Mathematically, it is also a very special number. The minimal length of doubly-even self-dual binary linear codes is 24. Indeed there is a unique such code of length 24 (cf. [P]), known as the binary Golay code (cf. [Go]). The automorphism group of this code is a sporadic finite simple group. The minimal dimension of even unimodular (self-dual) integral linear lattices without elements of square length 2 is also 24. Again there exists a unique such lattice of dimension 24 (cf. [C1]), known as Leech's lattice (cf. [Le]). Conway [C2] found three sporadic finite simple groups from the automorphism group of Leech's lattice. Griess [Gr] constructed the Monster, the largest sporadic finite simple group, as the automorphism group of a commutative non-associative algebra related to Leech's lattice. The Dedekind function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi zi} \quad (1.2)$$

is a fundamental modular form of weight $1/2$ in number theory, where $q^{1/24}$ is crucial for the modularity. Moreover, the Ramanujan series

$$\Delta_{24}(z) = (\eta(z))^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \quad (1.3)$$

where $\tau(n)$ is called the τ -function of Ramanujan. The function $\tau(n)$ is multiplicative and has nice congruence properties such as $\tau(7m+3) \equiv 0 \pmod{7}$ and $\tau(23+k) \equiv 0 \pmod{23}$ when k is any quadratic non-residue of 23. The theta series of an integral linear lattice is the generating function of counting the numbers of lattice points on spheres. Hecke [He] proved that the theta series of any even unimodular lattice must be a polynomial in the Eisenstein series $E_4(z)$ and the Ramanujan series $\Delta_{24}(z)$. The factor $(1 - q^n)^{24}$ is important to $\Delta_{24}(z)$. Let λ_r be the r th fundamental weight of \mathcal{G} . In this article, we

obtain the following identity

$$(1+q)(1+q+q^2) = (1-q)^{24} \sum_{m,n,k=0}^{\infty} (\dim V(m\lambda_3 + (n+k)\lambda_4)) q^{3m+2n+k} \quad (1.4)$$

when \mathcal{G} is the simple Lie algebra of type F_4 . In other words, the dimensions of the modules $V(k\lambda_3 + l\lambda_4)$ are linearly correlated by the binomial coefficients of 24. Numerically,

$$\begin{aligned} \dim V(k\lambda_3 + l\lambda_4) = & \frac{(l+1)(k+3)(k+l+4)}{39504568320000} (2k+l+7)(3k+l+10)(3k+2l+11) \\ & \times \left(\prod_{r=1}^5 (k+r) \right) \left(\prod_{s=2}^6 (k+l+s) \right) \left(\prod_{q=5}^9 (2k+l+q) \right). \end{aligned} \quad (1.5)$$

A direct elementary proof of (1.4) seems unthinkable.

The 52-dimensional exceptional simple Lie algebra \mathcal{G}^{F_4} of type F_4 can be realized as the full derivation algebra of the unique exceptional finite-dimensional simple Jordan algebra, which is 27-dimensional (e.g., cf. [A]). The identity element spans a one-dimensional trivial module. The quotient space of the Jordan algebra over the trivial module forms a 26-dimensional irreducible \mathcal{G}^{F_4} -module, which is the unique \mathcal{G}^{F_4} -module of minimal dimension (So it is called the *basic module of \mathcal{G}^{F_4}*). Singular (highest-weight) vectors of a representation of \mathcal{G}^{F_4} are weight vectors in the underlying module that are nullified by positive root vectors. In this article, we use partial partial differential equations to find all the singular vectors in the polynomial algebra over the basic irreducible module of \mathcal{G}^{F_4} . Then the identity (1.4) is a consequence of the Weyl's theorem of complete reducibility. Another corollary of our main theorem is that the algebra of polynomial invariants over the basic module is generated by two explicit invariants. In addition, there is also a simple application to harmonic analysis.

Denote by $E_{r,s}$ the square matrix with 1 as its (r,s) -entry and 0 as the others. The orthogonal Lie algebra

$$o(n, \mathbb{C}) = \sum_{1 \leq r < s \leq n} \mathbb{C}(E_{r,s} - E_{s,r}). \quad (1.6)$$

It acts on the polynomial algebra $\mathcal{A} = \mathbb{C}[x_1, \dots, x_n]$ by

$$(E_{r,s} - E_{s,r})|_{\mathcal{A}} = x_r \partial_{x_s} - x_s \partial_{x_r}. \quad (1.7)$$

Denote by \mathcal{A}_k the subspace of homogeneous polynomials in \mathcal{A} with degree k . When $n \geq 3$, it is well known that the subspace of harmonic polynomials

$$\mathcal{H}_k = \{f \in \mathcal{A}_k \mid (\partial_{x_1}^2 + \dots + \partial_{x_n}^2)(f) = 0\} \quad (1.8)$$

forms an irreducible $o(n, \mathbb{C})$ -module. The basic module of \mathcal{G}^{F_4} has an invariant bilinear form. So the subspace $\mathcal{H}_k^{F_4}$ of homogeneous harmonic polynomials over the basic module

with degree k (in different forms) also forms a finite-dimensional \mathcal{G}^{F_4} -submodule. According to the Weyl's theorem of complete reducibility, it is a direct sum of irreducible submodules. The subspace $\mathcal{H}_1^{F_4} = \mathcal{A}_1$ is the basic module itself. We deduce from our main theorem that the number of irreducible summands of $\mathcal{H}_k^{F_4}$ is $\geq \lfloor k/3 \rfloor + \lfloor (k-2)/3 \rfloor + 2$ for $k \geq 2$.

Our idea of using partial differential equations to solve Lie algebra problems started in our earlier works [X1] and [X2] when we tried to find functional generators for the invariants over curvature tensor fields and for the differential invariants of classical groups. Later we used partial differential equations to find the explicit formulas for all the singular vectors in the Verma modules of $sl(n, \mathbb{C})$ (cf. [X3]) and $sp(4, \mathbb{C})$ (cf. [X4], where the singular vectors related to Jantzen's work [J] for general $sp(2n, \mathbb{C})$ were also explicitly given). A few years ago, we realized that decomposing the polynomial algebra over a finite-dimensional module of a simple Lie algebra into a direct sum of irreducible submodules is equivalent to solving the differential equations of flag type:

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0, \quad (1.9)$$

where d_1, d_2, \dots, d_n are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ and f_1, \dots, f_{n-1} are polynomials satisfying

$$d_i(f_j) = 0 \quad \text{if } i > j. \quad (1.10)$$

In [X5], the methods of solving such equations were given. In particular, we found new special functions by which we are able to explicitly give the solutions of initial value problems of a large family of constant-coefficient linear partial differential equations in terms of their coefficients. Recently, Luo [Lu] used our methods to obtain explicit bases of certain infinite-dimensional non-canonical irreducible polynomial representations for classical simple Lie algebras. For convenience, we will use the notion

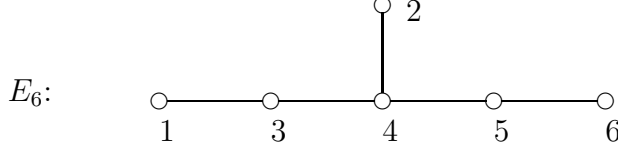
$$\overline{i, i+j} = \{i, i+1, i+2, \dots, i+j\} \quad (1.11)$$

for integer i and positive integer j throughout this article.

Since our proof of the main theorem heavily depends on precise explicit representation formulas, we present in Section 2 a construction of the basic representation of \mathcal{G}^{F_4} from the simple Lie algebra \mathcal{G}^{E_6} of type E_6 . In this way, the reader has the whole picture of our story, and it is easier for us to track errors. The proofs of our main theorem and its corollaries are given in Section 3

2 Basic Representation of F_4

We start with the root lattice construction of the simple Lie algebra of type E_6 . As we all known, the Dynkin diagram of E_6 is as follows:



Let $\{\alpha_i \mid i \in \overline{1,6}\}$ be the simple positive roots corresponding to the vertices in the diagram, and let Φ_{E_6} be the root system of E_6 . Set

$$Q_{E_6} = \sum_{i=1}^6 \mathbb{Z}\alpha_i, \quad (2.1)$$

the root lattice of type E_6 . Denote by (\cdot, \cdot) the symmetric \mathbb{Z} -bilinear form on Q_{E_6} such that

$$\Phi_{E_6} = \{\alpha \in Q_{E_6} \mid (\alpha, \alpha) = 2\}. \quad (2.2)$$

From the above Dynkin diagram of E_6 , we have the following automorphism of Q_{E_6} :

$$\sigma\left(\sum_{i=1}^6 k_i \alpha_i\right) = k_6 \alpha_1 + k_2 \alpha_2 + k_5 \alpha_3 + k_4 \alpha_4 + k_3 \alpha_5 + k_1 \alpha_6 \quad (2.3)$$

for $\sum_{i=1}^6 k_i \alpha_i \in Q_{E_6}$. Define a map $F : Q_{E_6} \times Q_{E_6} \rightarrow \{1, -1\}$ by

$$F\left(\sum_{i=1}^6 k_i \alpha_i, \sum_{j=1}^6 l_j \alpha_j\right) = (-1)^{\sum_{i=1}^6 k_i l_i + k_1 l_3 + k_4 l_2 + k_3 l_4 + k_5 l_4 + k_6 l_5}, \quad k_i, l_j \in \mathbb{Z}. \quad (2.4)$$

Then for $\alpha, \beta, \gamma \in Q_{E_6}$,

$$F(\alpha + \beta, \gamma) = F(\alpha, \gamma)F(\beta, \gamma), \quad F(\alpha, \beta + \gamma) = F(\alpha, \beta)F(\alpha, \gamma), \quad (2.5)$$

$$F(\alpha, \beta)F(\beta, \alpha)^{-1} = (-1)^{(\alpha, \beta)}, \quad F(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}. \quad (2.6)$$

In particular,

$$F(\alpha, \beta) = -F(\beta, \alpha) \quad \text{if } \alpha, \beta, \alpha + \beta \in \Phi_{E_6}. \quad (2.7)$$

Furthermore,

$$F(\sigma(\alpha), \sigma(\beta)) = F(\alpha, \beta) \quad \text{for } \alpha, \beta \in Q_{E_6}. \quad (2.8)$$

Denote

$$H = \bigoplus_{i=1}^6 \mathbb{C}\alpha_i. \quad (2.9)$$

Then the simple Lie algebra of type E_6 is

$$\mathcal{G}^{E_6} = H \oplus \bigoplus_{\alpha \in \Phi_{E_6}} \mathbb{C}E_\alpha \quad (2.10)$$

with the Lie bracket $[\cdot, \cdot]$ determined by:

$$[H, H] = 0, \quad [h, E_\alpha] = -[E_\alpha, h] = (h, \alpha)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = -\alpha, \quad (2.11)$$

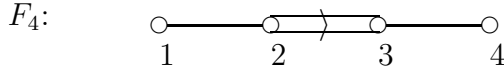
$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi_{E_6}, \\ F(\alpha, \beta)E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi_{E_6}. \end{cases} \quad (2.12)$$

Moreover, we have the following automorphism $\hat{\sigma}$ of the Lie algebra \mathcal{G}^{E_6} with order 2:

$$\hat{\sigma}\left(\sum_{i=1}^6 b_i \alpha_i\right) = \sum_{i=1}^6 b_i \sigma(\alpha_i), \quad b_i \in \mathbb{C}, \quad (2.13)$$

$$\hat{\sigma}(E_\alpha) = E_{\sigma(\alpha)} \quad \text{for } \alpha \in \Phi_{E_6}. \quad (2.14)$$

The Dynkin diagram of F_4 is



In order to make notation distinguishable, we add a bar on the roots in the root system Φ_{F_4} of type F_4 . In particular, we let $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4\}$ be the simple positive roots corresponding to the above Dynkin diagram of F_4 , where $\bar{\alpha}_1, \bar{\alpha}_2$ are long roots and $\bar{\alpha}_3, \bar{\alpha}_4$ are short roots. The simple Lie algebra of type F_4 is

$$\mathcal{G}^{F_4} = \{u \in \mathcal{G}^{E_6} \mid \hat{\sigma}(u) = u\} \quad (2.15)$$

with Cartan subalgebra

$$H_{F_4} = \mathbb{C}(\alpha_1 + \alpha_6) + \mathbb{C}(\alpha_3 + \alpha_5) + \mathbb{C}\alpha_4 + \mathbb{F}\alpha_2. \quad (2.16)$$

Set

$$V = \{w \in \mathcal{G}^{E_6} \mid \hat{\sigma}(w) = -w\}. \quad (2.17)$$

Then V forms the basic 26-dimensional \mathcal{G}^{F_4} -module with representation $\text{ad}|_{\mathcal{G}^{E_6}}$.

Next we want to find the explicit representation formulas of the root vectors of \mathcal{G}^{F_4} on V in terms of differential operators. Note the σ -invariant positive roots of Φ_{E_6} are:

$$\alpha_2, \alpha_4, \alpha_2 + \alpha_4, \sum_{i=3}^5 \alpha_i, \sum_{i=2}^5 \alpha_i, \alpha_1 + \sum_{i=3}^6 \alpha_i, \alpha_4 + \sum_{i=2}^5 \alpha_i, \sum_{i=1}^6 \alpha_i, \quad (2.18)$$

$$\alpha_4 + \sum_{i=1}^6 \alpha_i, \sum_{i=1}^6 \alpha_i + \sum_{r=3}^5 \alpha_r, \alpha_4 + \sum_{i=1}^6 \alpha_i + \sum_{r=3}^5 \alpha_r, \alpha_4 + \sum_{i=1}^6 \alpha_i + \sum_{r=2}^5 \alpha_r. \quad (2.19)$$

The followings are representatives of changing positive roots modulo σ :

$$\alpha_1, \alpha_3, \alpha_1 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \quad (2.20)$$

$$\sum_{i=1}^4 \alpha_i, \sum_{i=1}^5 \alpha_i, \alpha_4 + \sum_{i=1}^5 \alpha_i, \alpha_3 + \alpha_4 + \sum_{i=1}^5 \alpha_i, \alpha_3 + \alpha_4 + \sum_{i=1}^6 \alpha_i. \quad (2.21)$$

Set

$$x_1 = E_{\alpha_3 + \alpha_4 + \sum_{i=1}^6 \alpha_i} - E_{\alpha_5 + \alpha_4 + \sum_{i=1}^6 \alpha_i}, \quad x_2 = E_{\alpha_3 + \alpha_4 + \sum_{i=1}^5 \alpha_i} - E_{\alpha_4 + \alpha_5 + \sum_{i=2}^6 \alpha_i}, \quad (2.22)$$

$$x_3 = E_{\alpha_4 + \sum_{i=1}^5 \alpha_i} - E_{\alpha_4 + \sum_{i=2}^6 \alpha_i}, \quad x_4 = E_{\sum_{i=1}^5 \alpha_i} - E_{\sum_{i=2}^6 \alpha_i}, \quad (2.23)$$

$$x_5 = E_{\sum_{i=1}^4 \alpha_i} - E_{\alpha_2 + \sum_{i=4}^6 \alpha_i}, \quad x_6 = E_{\alpha_1 + \sum_{i=3}^5 \alpha_i} - E_{\sum_{i=3}^6 \alpha_i}, \quad (2.24)$$

$$x_7 = E_{\alpha_2 + \alpha_3 + \alpha_4} - E_{\alpha_2 + \alpha_4 + \alpha_5}, \quad x_8 = E_{\alpha_1 + \alpha_3 + \alpha_4} - E_{\sum_{i=4}^6 \alpha_i}, \quad x_9 = E_{\alpha_3 + \alpha_4} - E_{\alpha_4 + \alpha_5} \quad (2.25)$$

$$x_{10} = E_{\alpha_1 + \alpha_3} - E_{\alpha_5 + \alpha_6}, \quad x_{11} = E_{\alpha_3} - E_{\alpha_5}, \quad x_{12} = E_{\alpha_1} - E_{\alpha_6}, \quad (2.26)$$

$$x_{13} = \alpha_1 - \alpha_6, \quad x_{14} = \alpha_3 - \alpha_5, \quad x_{15} = E_{-\alpha_1} - E_{-\alpha_6}, \quad (2.27)$$

$$x_{16} = E_{-\alpha_3} - E_{-\alpha_5}, \quad x_{17} = E_{-\alpha_1 - \alpha_3} - E_{-\alpha_5 - \alpha_6}, \quad x_{18} = E_{-\alpha_3 - \alpha_4} - E_{-\alpha_4 - \alpha_5}, \quad (2.28)$$

$$x_{19} = E_{-\alpha_1 - \alpha_3 - \alpha_4} - E_{-\sum_{i=4}^6 \alpha_i}, \quad x_{20} = E_{-\alpha_2 - \alpha_3 - \alpha_4} - E_{-\alpha_2 - \alpha_4 - \alpha_5}, \quad (2.29)$$

$$x_{21} = E_{-\alpha_1 - \sum_{i=3}^5 \alpha_i} - E_{-\sum_{i=3}^6 \alpha_i}, \quad x_{22} = E_{-\sum_{i=1}^4 \alpha_i} - E_{-\alpha_2 - \sum_{i=4}^6 \alpha_i}, \quad (2.30)$$

$$x_{23} = E_{-\sum_{i=1}^5 \alpha_i} - E_{-\sum_{i=2}^6 \alpha_i}, \quad x_{25} = E_{-\alpha_3 - \alpha_4 - \sum_{i=1}^5 \alpha_i} - E_{-\alpha_4 - \alpha_5 - \sum_{i=2}^6 \alpha_i}, \quad (2.31)$$

$$x_{24} = E_{-\alpha_4 - \sum_{i=1}^5 \alpha_i} - E_{-\alpha_4 - \sum_{i=2}^6 \alpha_i}, \quad x_{26} = E_{-\alpha_3 - \alpha_4 - \sum_{i=1}^6 \alpha_i} - E_{-\alpha_4 - \alpha_5 - \sum_{i=1}^6 \alpha_i}. \quad (2.32)$$

Then the set $\{x_i \mid i \in \overline{1, 26}\}$ forms a basis of V and we treat all x_i as variables for technical convenience.

Again denote by $E_{\bar{\alpha}}$ the root vectors of \mathcal{G}^{F_4} as follows: $\varepsilon = \pm 1$,

$$E_{\varepsilon \bar{\alpha}_1} = E_{\varepsilon \alpha_2}, \quad E_{\varepsilon \bar{\alpha}_2} = E_{\varepsilon \alpha_4}, \quad E_{\varepsilon \bar{\alpha}_3} = E_{\varepsilon \alpha_3} + E_{\varepsilon \alpha_5}, \quad E_{\varepsilon \bar{\alpha}_4} = E_{\varepsilon \alpha_1} + E_{\varepsilon \alpha_6}, \quad (2.33)$$

$$E_{\varepsilon(\bar{\alpha}_1 + \bar{\alpha}_2)} = E_{\varepsilon(\alpha_2 + \alpha_4)}, \quad E_{\varepsilon(\bar{\alpha}_2 + \bar{\alpha}_3)} = E_{\varepsilon(\alpha_3 + \alpha_4)} + E_{\varepsilon(\alpha_4 + \alpha_5)}, \quad (2.34)$$

$$E_{\varepsilon(\bar{\alpha}_3 + \bar{\alpha}_4)} = E_{\varepsilon(\alpha_1 + \alpha_3)} + E_{\varepsilon(\alpha_5 + \alpha_6)}, \quad E_{\varepsilon(\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3)} = E_{\varepsilon(\alpha_2 + \alpha_3 + \alpha_4)} + E_{\varepsilon(\alpha_2 + \alpha_4 + \alpha_5)}, \quad (2.35)$$

$$E_{\varepsilon(\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4)} = E_{\varepsilon(\alpha_1 + \alpha_3 + \alpha_4)} + E_{\varepsilon(\alpha_4 + \alpha_5 + \alpha_6)}, \quad E_{\varepsilon(\bar{\alpha}_2 + 2\bar{\alpha}_3)} = E_{\varepsilon \sum_{i=3}^5 \alpha_i}, \quad (2.36)$$

$$E_{\varepsilon(\bar{\alpha}_1 + \bar{\alpha}_2 + 2\bar{\alpha}_3)} = E_{\varepsilon \sum_{i=2}^5 \alpha_i}, \quad E_{\varepsilon(\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4)} = E_{\varepsilon(\alpha_1 + \sum_{i=3}^5 \alpha_i)} + E_{\varepsilon \sum_{i=3}^6 \alpha_i}, \quad (2.37)$$

$$E_{\varepsilon \sum_{i=1}^4 \bar{\alpha}_i} = E_{\varepsilon(\sum_{i=1}^4 \alpha_i)} + E_{\varepsilon(\alpha_2 + \sum_{i=4}^6 \alpha_i)}, \quad E_{\varepsilon(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3)} = E_{\varepsilon(\alpha_4 + \sum_{i=2}^5 \alpha_i)}, \quad (2.38)$$

$$E_{\varepsilon \bar{\alpha}_3 + \varepsilon \sum_{i=1}^4 \bar{\alpha}_i} = E_{\varepsilon(\sum_{i=1}^5 \alpha_i)} + E_{\varepsilon(\sum_{i=2}^6 \alpha_i)}, \quad E_{\varepsilon(\bar{\alpha}_2 + 2\bar{\alpha}_3 + 2\bar{\alpha}_4)} = E_{\varepsilon(\alpha_1 + \sum_{i=3}^6 \alpha_i)} \quad (2.39)$$

$$E_{\varepsilon(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4)} = E_{\varepsilon(\alpha_4 + \sum_{i=1}^5 \alpha_i)} + E_{\varepsilon(\alpha_4 + \sum_{i=2}^6 \alpha_i)}, \quad (2.40)$$

$$E_{\varepsilon(\bar{\alpha}_1+\bar{\alpha}_2+2\bar{\alpha}_3+2\bar{\alpha}_4)} = E_{\varepsilon\sum_{i=1}^6\alpha_i}, \quad E_{\varepsilon(\bar{\alpha}_1+2\sum_{i=2}^4\bar{\alpha}_i)} = E_{\varepsilon(\alpha_4+\sum_{i=1}^6\alpha_i)}, \quad (2.41)$$

$$E_{\varepsilon(\bar{\alpha}_1+2\bar{\alpha}_2+3\bar{\alpha}_3+\bar{\alpha}_4)} = E_{\varepsilon(\alpha_3+\alpha_4+\sum_{i=1}^5\alpha_i)} + E_{\varepsilon(\alpha_4+\alpha_5+\sum_{i=2}^6\alpha_i)}, \quad (2.42)$$

$$E_{\varepsilon(\bar{\alpha}_1+2\bar{\alpha}_2+3\bar{\alpha}_3+2\bar{\alpha}_4)} = E_{\varepsilon(\alpha_3+\alpha_4+\sum_{i=1}^6\alpha_i)} + E_{\varepsilon(\alpha_4+\alpha_5+\sum_{i=1}^6\alpha_i)}, \quad (2.43)$$

$$E_{\varepsilon(\bar{\alpha}_1+2\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4)} = E_{\varepsilon(\sum_{i=1}^6\alpha_i+\sum_{r=3}^5\alpha_r)}, \quad (2.44)$$

$$E_{\varepsilon(\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4)} = E_{\varepsilon(\alpha_4+\sum_{i=1}^6\alpha_i+\sum_{r=3}^5\alpha_r)}, \quad (2.45)$$

$$E_{\varepsilon(2\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4)} = E_{\varepsilon(\alpha_4+\sum_{i=1}^6\alpha_i+\sum_{r=2}^5\alpha_r)}. \quad (2.46)$$

Moreover, we set

$$h_1 = \alpha_2, \quad h_2 = \alpha_4, \quad h_3 = \alpha_3 + \alpha_5, \quad h_4 = \alpha_1 + \alpha_6. \quad (2.47)$$

We calculate

$$E_{\bar{\alpha}_1}|_V = x_4\partial_{x_6} + x_5\partial_{x_8} + x_7\partial_{x_9} - x_{18}\partial_{x_{20}} - x_{19}\partial_{x_{22}} - x_{21}\partial_{x_{23}}, \quad (2.48)$$

$$E_{\bar{\alpha}_2}|_V = x_3\partial_{x_4} + x_8\partial_{x_{10}} + x_9\partial_{x_{11}} - x_{16}\partial_{x_{18}} - x_{17}\partial_{x_{19}} - x_{23}\partial_{x_{24}}, \quad (2.49)$$

$$\begin{aligned} E_{\bar{\alpha}_3}|_V &= -x_2\partial_{x_3} - x_4\partial_{x_5} - x_6\partial_{x_8} + x_{10}\partial_{x_{12}} + x_{11}(\partial_{x_{13}} - 2\partial_{x_{14}}) \\ &\quad - x_{14}\partial_{x_{16}} - x_{15}\partial_{x_{17}} + x_{19}\partial_{x_{21}} + x_{22}\partial_{x_{23}} + x_{24}\partial_{x_{25}}, \end{aligned} \quad (2.50)$$

$$\begin{aligned} E_{\bar{\alpha}_4}|_V &= -x_1\partial_{x_2} - x_5\partial_{x_7} - x_8\partial_{x_9} - x_{10}\partial_{x_{11}} + x_{12}(\partial_{x_{14}} - 2\partial_{x_{13}}) \\ &\quad - x_{13}\partial_{x_{15}} + x_{16}\partial_{x_{17}} + x_{18}\partial_{x_{19}} + x_{20}\partial_{x_{22}} + x_{25}\partial_{x_{26}}, \end{aligned} \quad (2.51)$$

$$E_{\bar{\alpha}_1+\bar{\alpha}_2}|_V = -x_3\partial_{x_6} + x_5\partial_{x_{10}} + x_7\partial_{x_{11}} - x_{16}\partial_{x_{20}} - x_{17}\partial_{x_{22}} + x_{21}\partial_{x_{24}}, \quad (2.52)$$

$$\begin{aligned} E_{\bar{\alpha}_2+\bar{\alpha}_3}|_V &= x_2\partial_{x_4} + x_3\partial_{x_5} + x_6\partial_{x_{10}} + x_8\partial_{x_{12}} + x_9(\partial_{x_{13}} - 2\partial_{x_{14}}) \\ &\quad - x_{14}\partial_{x_{18}} - x_{15}\partial_{x_{19}} - x_{17}\partial_{x_{21}} - x_{22}\partial_{x_{24}} - x_{23}\partial_{x_{25}}, \end{aligned} \quad (2.53)$$

$$\begin{aligned} E_{\bar{\alpha}_3+\bar{\alpha}_4}|_V &= -x_1\partial_{x_3} + x_4\partial_{x_7} + x_6\partial_{x_9} - x_{10}(\partial_{x_{13}} + \partial_{x_{14}}) - x_{11}\partial_{x_{15}} \\ &\quad + x_{12}\partial_{x_{16}} - (x_{13} + x_{14})\partial_{x_{17}} - x_{18}\partial_{x_{21}} - x_{20}\partial_{x_{23}} + x_{24}\partial_{x_{26}}, \end{aligned} \quad (2.54)$$

$$\begin{aligned} E_{\bar{\alpha}_1+\bar{\alpha}_2+\bar{\alpha}_3}|_V &= -x_2\partial_{x_6} + x_3\partial_{x_8} + x_4\partial_{x_{10}} + x_5\partial_{x_{12}} + x_7(\partial_{x_{13}} - 2\partial_{x_{14}}) - x_{14}\partial_{x_{20}} \\ &\quad - x_{15}\partial_{x_{22}} - x_{17}\partial_{x_{23}} - x_{19}\partial_{x_{24}} + x_{21}\partial_{x_{25}}, \end{aligned} \quad (2.55)$$

$$\begin{aligned} E_{\bar{\alpha}_2+\bar{\alpha}_3+\bar{\alpha}_4}|_V &= x_1\partial_{x_4} + x_3\partial_{x_7} - x_6\partial_{x_{11}} - x_8(\partial_{x_{13}} + \partial_{x_{14}}) - x_9\partial_{x_{15}} + x_{12}\partial_{x_{18}} \\ &\quad - (x_{13} + x_{14})\partial_{x_{19}} + x_{16}\partial_{x_{21}} - x_{20}\partial_{x_{24}} - x_{23}\partial_{x_{26}}, \end{aligned} \quad (2.56)$$

$$E_{\bar{\alpha}_2+2\bar{\alpha}_3}|_V = -x_2\partial_{x_5} + x_6\partial_{x_{12}} + x_9\partial_{x_{16}} - x_{11}\partial_{x_{18}} - x_{15}\partial_{x_{21}} + x_{22}\partial_{x_{25}}, \quad (2.57)$$

$$E_{\bar{\alpha}_1+\bar{\alpha}_2+2\bar{\alpha}_3}|_V = x_2\partial_{x_8} + x_4\partial_{x_{12}} + x_7\partial_{x_{16}} - x_{11}\partial_{x_{20}} - x_{15}\partial_{x_{23}} - x_{19}\partial_{x_{25}}, \quad (2.58)$$

$$\begin{aligned} E_{\bar{\alpha}_2+2\bar{\alpha}_3+\bar{\alpha}_4}|_V &= -x_1\partial_{x_5} + x_2\partial_{x_7} + x_6(\partial_{x_{14}} - 2\partial_{x_{13}}) + x_8\partial_{x_{16}} + x_9\partial_{x_{17}} \\ &\quad - x_{10}\partial_{x_{18}} - x_{11}\partial_{x_{19}} - x_{13}\partial_{x_{21}} - x_{20}\partial_{x_{25}} + x_{22}\partial_{x_{26}}, \end{aligned} \quad (2.59)$$

$$\begin{aligned} E_{\bar{\alpha}_1+\bar{\alpha}_2+\bar{\alpha}_3+\bar{\alpha}_4}|_V &= -x_1\partial_{x_6} - x_3\partial_{x_9} - x_4\partial_{x_{11}} - x_5(\partial_{x_{13}} + \partial_{x_{14}}) - x_7\partial_{x_{15}} + x_{12}\partial_{x_{20}} \\ &\quad - (x_{13} + x_{14})\partial_{x_{22}} + x_{16}\partial_{x_{23}} + x_{18}\partial_{x_{24}} + x_{21}\partial_{x_{26}}, \end{aligned} \quad (2.60)$$

$$E_{\bar{\alpha}_1+2\bar{\alpha}_2+2\bar{\alpha}_3}|_V = -x_2\partial_{x_{10}} + x_3\partial_{x_{12}} + x_7\partial_{x_{18}} - x_9\partial_{x_{20}} - x_{15}\partial_{x_{24}} + x_{17}\partial_{x_{25}}, \quad (2.61)$$

$$\begin{aligned} E_{\bar{\alpha}_1+\bar{\alpha}_2+2\bar{\alpha}_3+\bar{\alpha}_4}|_V &= x_1\partial_{x_8} - x_2\partial_{x_9} + x_4(\partial_{x_{14}} - 2\partial_{x_{13}}) + x_5\partial_{x_{16}} + x_7\partial_{x_{17}} - x_{10}\partial_{x_{20}} \\ &\quad - x_{11}\partial_{x_{22}} - x_{13}\partial_{x_{23}} + x_{18}\partial_{x_{25}} - x_{19}\partial_{x_{26}}, \end{aligned} \quad (2.62)$$

$$E_{\bar{\alpha}_2+2\bar{\alpha}_3+2\bar{\alpha}_4}|_V = x_1\partial_{x_7} + x_6\partial_{x_{15}} + x_8\partial_{x_{17}} - x_{10}\partial_{x_{19}} - x_{12}\partial_{x_{21}} - x_{20}\partial_{x_{26}}, \quad (2.63)$$

$$\begin{aligned} E_{\bar{\alpha}_1+2\bar{\alpha}_2+2\bar{\alpha}_3+\bar{\alpha}_4}|_V &= -x_1\partial_{x_{10}} + x_2\partial_{x_{11}} + x_3(\partial_{x_{14}} - 2\partial_{x_{13}}) + x_5\partial_{x_{18}} + x_7\partial_{x_{19}} - x_8\partial_{x_{20}} \\ &\quad - x_9\partial_{x_{22}} - x_{13}\partial_{x_{24}} - x_{16}\partial_{x_{25}} + x_{17}\partial_{x_{26}}, \end{aligned} \quad (2.64)$$

$$E_{\bar{\alpha}_1+\bar{\alpha}_2+2\bar{\alpha}_3+2\bar{\alpha}_4}|_V = -x_1\partial_{x_9} + x_4\partial_{x_{15}} + x_5\partial_{x_{17}} - x_{10}\partial_{x_{22}} - x_{12}\partial_{x_{23}} + x_{18}\partial_{x_{26}}, \quad (2.65)$$

$$E_{\bar{\alpha}_1+2\bar{\alpha}_2+2\bar{\alpha}_3+2\bar{\alpha}_4}|_V = x_1\partial_{x_{11}} + x_3\partial_{x_{15}} + x_5\partial_{x_{19}} - x_8\partial_{x_{22}} - x_{12}\partial_{x_{24}} - x_{16}\partial_{x_{26}}, \quad (2.66)$$

$$\begin{aligned} E_{\bar{\alpha}_1+2\bar{\alpha}_2+3\bar{\alpha}_3+\bar{\alpha}_4}|_V &= -x_1\partial_{x_{12}} - x_2(\partial_{x_{14}} + \partial_{x_{13}}) - x_3\partial_{x_{16}} + x_4\partial_{x_{18}} - x_6\partial_{x_{20}} + x_7\partial_{x_{21}} \\ &\quad - x_9\partial_{x_{23}} + x_{11}\partial_{x_{24}} - (x_{13} + x_{14})\partial_{x_{25}} + x_{15}\partial_{x_{26}}, \end{aligned} \quad (2.67)$$

$$\begin{aligned} E_{\bar{\alpha}_1+2\bar{\alpha}_2+3\bar{\alpha}_3+2\bar{\alpha}_4}|_V &= x_1(\partial_{x_{13}} - 2\partial_{x_{14}}) + x_2\partial_{x_{15}} - x_3\partial_{x_{17}} + x_4\partial_{x_{19}} + x_5\partial_{x_{21}} - x_6\partial_{x_{22}} \\ &\quad - x_8\partial_{x_{23}} + x_{10}\partial_{x_{24}} - x_{12}\partial_{x_{25}} - x_{14}\partial_{x_{26}}, \end{aligned} \quad (2.68)$$

$$E_{\bar{\alpha}_1+2\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}|_V = x_1\partial_{x_{16}} - x_2\partial_{x_{17}} + x_4\partial_{x_{21}} - x_6\partial_{x_{23}} + x_{10}\partial_{x_{25}} - x_{11}\partial_{x_{26}}, \quad (2.69)$$

$$E_{\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}|_V = -x_1\partial_{x_{18}} + x_2\partial_{x_{19}} - x_3\partial_{x_{21}} + x_6\partial_{x_{24}} - x_8\partial_{x_{25}} + x_9\partial_{x_{26}}, \quad (2.70)$$

$$E_{2\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}|_V = x_1\partial_{x_{20}} - x_2\partial_{x_{22}} + x_3\partial_{x_{23}} - x_4\partial_{x_{24}} + x_5\partial_{x_{25}} - x_7\partial_{x_{26}}, \quad (2.71)$$

$$\begin{aligned} h_1|_V &= x_4\partial_{x_4} + x_5\partial_{x_5} - x_6\partial_{x_6} + x_7\partial_{x_7} - x_8\partial_{x_8} - x_9\partial_{x_9} + x_{18}\partial_{x_{18}} \\ &\quad + x_{19}\partial_{x_{19}} - x_{20}\partial_{x_{20}} + x_{21}\partial_{x_{21}} - x_{22}\partial_{x_{22}} - x_{23}\partial_{x_{23}}, \end{aligned} \quad (2.72)$$

$$\begin{aligned}
h_2|_V &= x_3\partial_{x_3} - x_4\partial_{x_4} + x_8\partial_{x_8} + x_9\partial_{x_9} - x_{10}\partial_{x_{10}} - x_{11}\partial_{x_{11}} + x_{16}\partial_{x_{16}} \\
&\quad + x_{17}\partial_{x_{17}} - x_{18}\partial_{x_{18}} - x_{19}\partial_{x_{19}} + x_{23}\partial_{x_{23}} - x_{24}\partial_{x_{24}},
\end{aligned} \tag{2.73}$$

$$\begin{aligned}
h_3|_V &= x_2\partial_{x_2} - x_3\partial_{x_3} + x_4\partial_{x_4} - x_5\partial_{x_5} + x_6\partial_{x_6} - x_8\partial_{x_8} + x_{10}\partial_{x_{10}} \\
&\quad + 2x_{11}\partial_{x_{11}} - x_{12}\partial_{x_{12}} + x_{15}\partial_{x_{15}} - 2x_{16}\partial_{x_{16}} - x_{17}\partial_{x_{17}} + x_{19}\partial_{x_{19}} \\
&\quad - x_{21}\partial_{x_{21}} + x_{22}\partial_{x_{22}} - x_{23}\partial_{x_{23}} + x_{24}\partial_{x_{24}} - x_{25}\partial_{x_{25}},
\end{aligned} \tag{2.74}$$

$$\begin{aligned}
h_4|_V &= x_1\partial_{x_1} - x_2\partial_{x_2} + x_5\partial_{x_5} - x_7\partial_{x_7} + x_8\partial_{x_8} - x_9\partial_{x_9} + x_{10}\partial_{x_{10}} \\
&\quad - x_{11}\partial_{x_{11}} + 2x_{12}\partial_{x_{12}} - 2x_{15}\partial_{x_{15}} + x_{16}\partial_{x_{16}} - x_{17}\partial_{x_{17}} + x_{18}\partial_{x_{18}} \\
&\quad - x_{19}\partial_{x_{19}} + x_{20}\partial_{x_{20}} - x_{22}\partial_{x_{22}} + x_{25}\partial_{x_{25}} - x_{26}\partial_{x_{26}}.
\end{aligned} \tag{2.75}$$

Denote

$$\bar{r} = 27 - r \quad \text{for } r \in \overline{1, 26}. \tag{2.76}$$

Define an algebraic isomorphism τ on $\mathbb{C}[x_1, \dots, x_{26}][\partial_{x_1}, \dots, \partial_{x_{26}}]$ by

$$\tau(x_r) = x_{\bar{r}}, \quad \tau(\partial_{x_r}) = \partial_{x_{\bar{r}}} \quad \text{for } r \in \overline{1, 26} \setminus \{13, 14\} \tag{2.77}$$

and

$$\tau(x_{13}) = -x_{13}, \quad \tau(x_{14}) = -x_{14}, \quad \tau(\partial_{x_{13}}) = -\partial_{x_{13}}, \quad \tau(\partial_{x_{14}}) = -\partial_{x_{14}}. \tag{2.78}$$

Then

$$E_{-\bar{\alpha}}|_V = \tau(E_{\bar{\alpha}}|_V) \tag{2.79}$$

for $\bar{\alpha} \in \Phi_{F_4}^+$, the set of positive roots of \mathcal{G}^{F_4} . Thus we have given the explicit formulas for the basic irreducible representation of F_4 .

3 Polynomial Representation of F_4

According to (2.48)-(2.75), V is the irreducible module of the highest weight λ_4 and x_1 is a highest weight vector. In this section, we want to study the \mathcal{G}^{F_4} -module $\mathcal{A} = \mathbb{C}[x_i \mid i \in \overline{1, 26}]$ via the representation formulas (2.48)-(2.79).

Suppose that

$$\eta_1 = 3 \sum_{r=1}^{12} x_r x_{\bar{r}} + a x_{13}^2 + b x_{13} x_{14} + c x_{14}^2 \tag{3.1}$$

is an F_4 -invariant (cf. (2.76)), where a, b, c are constants to be determined. Then $E_{\bar{\alpha}_1}(\eta_1) = E_{\bar{\alpha}_2}(\eta_1) = 0$ naturally hold. Moreover,

$$0 = E_{\bar{\alpha}_3}(\eta_1) = 2(a - b)x_{11}x_{13} + [b - 4c - 3]x_{11}x_{14}, \tag{3.2}$$

which gives

$$a = b = 4c + 3, \quad . \quad (3.3)$$

Similarly, the constraint $E_{\bar{\alpha}_3}(\eta_1) = 0$ yield $b = c = 4a + 3$. So we have the quadratic invariant

$$\eta_1 = 3 \sum_{r=1}^{12} x_r x_{\bar{r}} - x_{13}^2 - x_{13}x_{14} - x_{14}^2. \quad (3.4)$$

This invariant also gives a symmetric \mathcal{G}^{F_4} -invariant bilinear form on V .

By (2.72)-(2.75), we try to find a quadratic singular vector of the form

$$\zeta_1 = x_1(a_1x_{13} + a_2x_{14}) + a_3x_2x_{12} + a_4x_3x_{10} + a_5x_4x_8 + a_6x_5x_6, \quad (3.5)$$

where a_r are constants. Observe

$$0 = E_{\bar{\alpha}_1}(\zeta_1) = (a_5 + a_6)x_4x_5 \implies a_6 = -a_5. \quad (3.5)$$

Moreover,

$$0 = E_{\bar{\alpha}_2}(\zeta_1) = (a_4 + a_5)x_3x_8 \implies a_5 = -a_4. \quad (3.6)$$

Note

$$0 = E_{\bar{\alpha}_3}(\zeta_1) = (a_1 - 2a_2)x_1x_{11} + (a_3 - a_4)x_2x_{10} \implies a_1 = 2a_2, \quad a_3 = a_4. \quad (3.7)$$

Furthermore,

$$0 = E_{\bar{\alpha}_4}(\zeta_1) = (a_2 - 2a_1 - a_3)x_1x_{12} \implies 2a_1 = a_2 - a_3. \quad (3.8)$$

Hence we have the singular vector

$$\zeta_1 = x_1(2x_{13} + x_{14}) - 3x_2x_{12} - 3x_3x_{10} + 3x_4x_8 - 3x_5x_6 \quad (3.9)$$

of weight λ_4 . So it generates an irreducible module that is isomorphic to the basic module V . Note

$$E_{-\bar{\alpha}_1}|_V = -x_6\partial_{x_4} - x_8\partial_{x_5} - x_9\partial_{x_7} + x_{20}\partial_{x_{18}} + x_{22}\partial_{x_{19}} + x_{23}\partial_{x_{21}}, \quad (3.10)$$

$$E_{-\bar{\alpha}_2}|_V = -x_4\partial_{x_3} - x_{10}\partial_{x_8} - x_{11}\partial_{x_9} + x_{18}\partial_{x_{16}} + x_{19}\partial_{x_{17}} + x_{24}\partial_{x_{23}}, \quad (3.11)$$

$$\begin{aligned} E_{-\bar{\alpha}_3}|_V &= x_3\partial_{x_2} + x_5\partial_{x_4} + x_8\partial_{x_6} - x_{12}\partial_{x_{10}} + x_{16}(2\partial_{x_{14}} - \partial_{x_{13}}) \\ &\quad + x_{14}\partial_{x_{11}} + x_{17}\partial_{x_{15}} - x_{21}\partial_{x_{19}} - x_{23}\partial_{x_{22}} - x_{25}\partial_{x_{24}}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} E_{-\bar{\alpha}_4}|_V &= x_2\partial_{x_1} + x_7\partial_{x_5} + x_9\partial_{x_8} + x_{11}\partial_{x_{10}} + x_{15}(2\partial_{x_{13}} - \partial_{x_{14}}) \\ &\quad + x_{13}\partial_{x_{12}} - x_{17}\partial_{x_{16}} - x_{19}\partial_{x_{18}} - x_{22}\partial_{x_{20}} - x_{26}\partial_{x_{25}} \end{aligned} \quad (3.13)$$

by (2.77)-(2.79). To get a basis of the module generated by ζ_1 compatible to $\{x_i \mid i \in \overline{1, 26}\}$, we set

$$\zeta_2 = E_{-\bar{\alpha}_4}(\zeta_1) = x_2(-x_{13} + x_{14}) + 3x_1x_{15} - 3x_3x_{11} + 3x_4x_9 - 3x_6x_7, \quad (3.14)$$

$$\zeta_3 = E_{-\bar{\alpha}_3}(\zeta_2) = -x_3(x_{13} + 2x_{14}) + 3x_1x_{17} + 3x_2x_{16} + 3x_5x_9 - 3x_7x_8, \quad (3.15)$$

$$\zeta_4 = -E_{-\bar{\alpha}_2}(\zeta_3) = -x_4(x_{13} + 2x_{14}) - 3x_1x_{19} - 3x_2x_{18} + 3x_5x_{11} - 3x_7x_{10}, \quad (3.16)$$

$$\zeta_5 = E_{-\bar{\alpha}_3}(\zeta_4) = x_5(-x_{13} + x_{14}) + 3x_1x_{21} - 3x_3x_{18} - 3x_4x_{16} + 3x_7x_{12}, \quad (3.17)$$

$$\zeta_6 = -E_{-\bar{\alpha}_1}(\zeta_4) = -x_6(x_{13} + 2x_{14}) + 3x_1x_{22} + 3x_2x_{20} + 3x_8x_{11} - 3x_9x_{10}, \quad (3.18)$$

$$\zeta_7 = E_{-\bar{\alpha}_4}(\zeta_5) = x_7(2x_{13} + x_{14}) + 3x_2x_{21} + 3x_3x_{19} + 3x_4x_{17} - 3x_5x_{15}, \quad (3.19)$$

$$\zeta_8 = -E_{-\bar{\alpha}_1}(\zeta_5) = x_8(-x_{13} + x_{14}) - 3x_1x_{23} + 3x_3x_{20} - 3x_6x_{16} + 3x_9x_{12}, \quad (3.20)$$

$$\zeta_9 = -E_{-\bar{\alpha}_1}(\zeta_7) = x_9(2x_{13} + x_{14}) - 3x_2x_{23} - 3x_3x_{22} + 3x_6x_{17} - 3x_8x_{15}, \quad (3.21)$$

$$\zeta_{10} = -E_{-\bar{\alpha}_2}(\zeta_8) = x_{10}(-x_{13} + x_{14}) + 3x_1x_{24} + 3x_4x_{20} + 3x_6x_{18} + 3x_{11}x_{12}, \quad (3.22)$$

$$\zeta_{11} = -E_{-\bar{\alpha}_2}(\zeta_9) = x_{11}(2x_{13} + x_{14}) + 3x_2x_{24} - 3x_4x_{22} - 3x_6x_{19} - 3x_{10}x_{15}, \quad (3.23)$$

$$\zeta_{12} = -E_{-\bar{\alpha}_3}(\zeta_{10}) = -x_{12}(x_{13} + 2x_{14}) + 3x_1x_{25} - 3x_5x_{20} - 3x_8x_{18} - 3x_{10}x_{16}, \quad (3.24)$$

$$\begin{aligned} \zeta_{13} = E_{-\bar{\alpha}_4}(\zeta_{12}) &= -x_{13}(x_{13} + 2x_{14}) - 3x_1x_{26} + 3x_2x_{25} + 3x_5x_{22} - 3x_7x_{20} \\ &\quad + 3x_8x_{19} - 3x_9x_{18} + 3x_{10}x_{17} - 3x_{11}x_{16}, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \zeta_{14} = E_{-\bar{\alpha}_3}(\zeta_{11}) &= x_{14}(2x_{13} + x_{14}) - 3x_2x_{25} + 3x_3x_{24} + 3x_4x_{23} - 3x_5x_{22} \\ &\quad + 3x_6x_{21} - 3x_8x_{19} - 3x_{10}x_{17} + 3x_{12}x_{15}, \end{aligned} \quad (3.26)$$

$$\zeta_r = \tau(\zeta_{\bar{r}}) \quad \text{for } r \in \overline{15, 27}, \quad (3.27)$$

where τ is an algebra automorphism determined by (2.77) and (2.78). The above construction shows that the map $x_r \mapsto \zeta_r$ determine a module isomorphism from V to the module generated by ζ_1 . In particular,

$$E_{\bar{\alpha}}(x_i) = ax_j \Leftrightarrow E_{\bar{\alpha}}(\zeta_i) = a\zeta_j, \quad a \in \mathbb{C}, \bar{\alpha} \in \Phi_{F_4}. \quad (3.28)$$

First

$$\begin{aligned} \vartheta = (x_1\zeta_2 - x_2\zeta_1)/3 &= x_1(-x_2x_{13} + x_1x_{15} - x_3x_{11} + x_4x_9 - x_6x_7) \\ &\quad + x_2(x_2x_{12} + x_3x_{10} - x_4x_8 + x_5x_6) \end{aligned} \quad (3.29)$$

is a singular vector of weight λ_3 . Recall that the invariant η_1 in (3.4) define an invariant bilinear form on V . Thus we have the following cubic invariant

$$\eta_2 = 3 \sum_{r=1}^{12} (\zeta_r x_{\bar{r}} + x_r \zeta_{\bar{r}}) - 2x_{13}\zeta_{13} - x_{13}\zeta_{14} - x_{14}\zeta_{13} - 2x_{14}\zeta_{14}. \quad (3.30)$$

According to (3.9) and (3.14)-(3.27), we find

$$\begin{aligned} \eta_2 = & 9(1 + \tau)[(x_2x_{12} + x_3x_{10} - x_4x_8 + x_5x_6)x_{26} + (x_3x_{11} - x_4x_9 + x_6x_7)x_{25} \\ & + (x_7x_8 - x_5x_9)x_{24} + x_{10}(x_4x_{23} + x_9x_{21}) - x_{11}(x_5x_{23} + x_8x_{21} + x_{12}x_{17}) \\ & - x_{12}(x_7x_{22} + x_9x_{19})] + 2x_{13}^3 + 3x_{13}^2x_{14} - 3x_{13}x_{14}^2 - 2x_{14}^3 + 3x_1(2x_{13} + x_{14})x_{26} \\ & + 3x_2(x_{13} + 2x_{14})x_{25} - 3x_{13}[x_3x_{24} + x_4x_{23} + x_5x_{22} + x_6x_{21} - 2x_7x_{20} + x_8x_{19} \\ & - 2x_9x_{18} + x_{10}x_{17} - 2x_{11}x_{16} + x_{12}x_{15}] - 3x_{14}[2x_3x_{24} + 2x_4x_{23} - x_5x_{22} \\ & + 2x_6x_{21} - x_7x_{20} - x_8x_{19} - x_9x_{18} - x_{10}x_{17} - x_{11}x_{16} + 2x_{12}x_{15}], \end{aligned} \quad (3.31)$$

where τ is an algebra automorphism defined in (2.77) and (2.78). Denote by \mathbb{N} the set of nonnegative integers. Now we are ready to prove our main theorem.

Theorem 3.1. *Any polynomial f in \mathcal{A} satisfying the system of partial differential equations*

$$E_{\bar{\alpha}}(f) = 0 \quad \text{for } \bar{\alpha} \in \Phi_{F_4}^+ \quad (3.32)$$

must be a polynomial in $x_1, \zeta_1, \vartheta, \eta_1, \eta_2$. In particular, the elements

$$\{x_1^{m_1}\zeta_1^{m_2}\vartheta^{m_3}\eta_1^{m_4}\eta_2^{m_5} \mid m_1, m_2, m_3, m_4, m_5 \in \mathbb{N}\} \quad (3.33)$$

are linearly independent singular vectors and any singular vector is a linear combination of those in (3.33) with the same weight. The weight of $x_1^{m_1}\zeta_1^{m_2}\vartheta^{m_3}\eta_1^{m_4}\eta_2^{m_5}$ is $m_3\lambda_3 + (m_1 + m_2)\lambda_4$.

Proof. First we note

$$x_1x_{14} = \zeta_1 - 2x_1x_{13} + 3x_2x_{12} + 3x_3x_{10} - 3x_4x_8 + 3x_5x_6, \quad (3.34)$$

$$3x_1x_{15} = \zeta_2 + x_2(x_{13} - x_{14}) + 3x_3x_{11} - 3x_4x_9 + 3x_6x_7, \quad (3.35)$$

$$3x_1x_{17} = \zeta_3 + x_3(x_{13} + 2x_{14}) - 3x_2x_{16} - 3x_5x_9 + 3x_7x_8, \quad (3.36)$$

$$3x_1x_{19} = 3x_5x_{11} - \zeta_4 - x_4(x_{13} + 2x_{14}) - 3x_2x_{18} - 3x_7x_{10}, \quad (3.37)$$

$$3x_1x_{21} = \zeta_5 + x_5(x_{13} - x_{14}) + 3x_3x_{18} + 3x_4x_{16} - 3x_7x_{12}, \quad (3.38)$$

$$3x_1x_{22} = \zeta_6 + x_6(x_{13} + 2x_{14}) - 3x_2x_{20} - 3x_8x_{11} + 3x_9x_{10}, \quad (3.39)$$

$$3x_1x_{23} = \zeta_8 + x_8(x_{13} - x_{14}) - 3x_3x_{20} + 3x_6x_{16} - 3x_9x_{12}, \quad (3.40)$$

$$3x_1x_{24} = \zeta_{10} + x_{10}(x_{13} - x_{14}) - 3x_4x_{20} - 3x_6x_{18} - 3x_{11}x_{12}, \quad (3.41)$$

$$3x_2x_{25} + 3x_1x_{26} = \eta_1 - 3 \sum_{r=3}^{12} x_r x_{\bar{r}} + x_{13}^2 + x_{13}x_{14} + x_{14}^2, \quad (3.42)$$

$$\begin{aligned} & 3[3(x_1x_{15} + x_3x_{11} - x_4x_9 + x_6x_7) + x_2(x_{13} + 2x_{14})]x_{25} \\ & + 3[3(x_2x_{12} + x_3x_{10} - x_4x_8 + x_5x_6) + x_1(2x_{13} + x_{14})]x_{26} \\ = & \eta_2 - 9x_1(x_{17}x_{24} - x_{19}x_{23} + x_{21}x_{22}) - 9x_2(x_{16}x_{24} - x_{18}x_{23} + x_{20}x_{21}) - 3x_{13}^2x_{14} \\ & - 9(1 + \tau)[(x_7x_8 - x_5x_9)x_{24} + x_{10}(x_4x_{23} + x_9x_{21}) - x_{11}(x_5x_{23} + x_8x_{21} + x_{12}x_{17}) \\ & - x_{12}(x_7x_{22} + x_9x_{19})] - 2x_{13}^3 + 3x_{13}x_{14}^2 + 2x_{14}^3 + 3x_{13}[x_3x_{24} + x_4x_{23} + x_5x_{22} + x_6x_{21} \\ & - 2x_7x_{20} + x_8x_{19} - 2x_9x_{18} + x_{10}x_{17} - 2x_{11}x_{16} + x_{12}x_{15}] - 3x_{14}[2x_3x_{24} + 2x_4x_{23} \\ & - x_5x_{22} + 2x_6x_{21} - x_7x_{20} - x_8x_{19} - x_9x_{18} - x_{10}x_{17} - x_{11}x_{16} + 2x_{12}x_{15}] \end{aligned} \quad (3.43)$$

by (3.4), (3.9), (3.14)-(3.18), (3.20), (3.22) and (3.31). Thus $\{x_r \mid 16, 18, 20 \neq r \in \overline{14, 26}\}$ are rational functions in

$$\{x_r, \zeta_s, \eta_1, \eta_2 \mid r \in \{\overline{1, 13}, 16, 18, 20\}; 7, 9 \neq s \in \overline{1, 10}\}. \quad (3.44)$$

Suppose that $f \in \mathcal{A}$ is a solution of (3.32). Write f as a rational function f_1 in the variables of (3.44). In the following calculations, we will always use (3.28). By (2.71),

$$0 = E_{2\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}(f_1) = x_1\partial_{x_{20}}(f_1). \quad (3.45)$$

So f_1 is independent of x_{20} . Moreover, (2.70) gives

$$0 = E_{\bar{\alpha}_1+3\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}(f_1) = -x_1\partial_{x_{18}}(f_1). \quad (3.46)$$

Hence f_1 is independent of x_{18} . Furthermore, (2.69) yields

$$0 = E_{\bar{\alpha}_1+2\bar{\alpha}_2+4\bar{\alpha}_3+2\bar{\alpha}_4}(f_1) = x_1\partial_{x_{16}}(f_1). \quad (3.47)$$

Thus f_1 is independent of x_{16} . Successively applying (2.68), (2.67), (2.66), (2.65) and (2.63) to f_1 , we obtain that f_1 is independent of $x_{13}, x_{12}, x_{11}, x_9$ and x_7 . Therefore, f_1 is a rational function in

$$\{x_r, \zeta_s, \eta_1, \eta_2 \mid 7, 9 \neq r \in \overline{1, 10}; 7, 9 \neq s \in \overline{1, 10}\}. \quad (3.48)$$

By (2.54), (2.56), (2.59), (2.60), (2.62) and (2.64),

$$0 = E_{\bar{\alpha}_3+\bar{\alpha}_4}(f_1) = -x_1\partial_{x_3}(f_1) - \zeta_1\partial_{\zeta_3}(f_1), \quad (3.49)$$

$$0 = E_{\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4}(f_1) = x_1 \partial_{x_4}(f_1) + \zeta_1 \partial_{\zeta_4}(f_1), \quad (3.50)$$

$$0 = E_{\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4}(f_1) = -x_1 \partial_{x_5}(f_1) - \zeta_1 \partial_{\zeta_5}(f_1), \quad (3.51)$$

$$0 = E_{\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4}(f_1) = -x_1 \partial_{x_6}(f_1) - \zeta_1 \partial_{\zeta_6}(f_1), \quad (3.52)$$

$$0 = E_{\bar{\alpha}_1 + \bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4}(f_1) = x_1 \partial_{x_8}(f_1) + \zeta_1 \partial_{\zeta_8}(f_1), \quad (3.53)$$

$$0 = E_{\bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4}(f_1) = -x_1 \partial_{x_{10}}(f_1) - \zeta_1 \partial_{\zeta_{10}}(f_1). \quad (3.54)$$

Set

$$\eta_r = x_1 \zeta_r - x_r \zeta_1, \quad r \in \overline{3, 6}; \quad \eta_7 = x_1 \zeta_8 - x_8 \zeta_1, \quad \eta_8 = x_1 \zeta_{10} - x_{10} \zeta_1. \quad (3.55)$$

By the characteristic method of solving linear partial differential equations, we get that f_1 can be written as a rational function f_2 in

$$\{x_r, \zeta_s, \eta_q \mid r, s = 1, 2; q \in \overline{1, 8}\}. \quad (3.56)$$

Next applying (2.57), (2.58) and (2.61) to f_2 , we get

$$0 = E_{\bar{\alpha}_2 + 2\bar{\alpha}_3}(f_2) = -(x_1 \zeta_2 - \zeta_1 x_2) \partial_{\eta_5}(f_2) = -3\vartheta \partial_{\eta_5}(f_2), \quad (3.57)$$

$$0 = E_{\bar{\alpha}_1 + \bar{\alpha}_2 + 2\bar{\alpha}_3}(f_2) = 3\vartheta \partial_{\eta_7}(f_2), \quad 0 = E_{\bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3}(f_2) = -3\vartheta \partial_{\eta_8}(f_2) = 0 \quad (3.58)$$

(cf. (3.29)). Thus f_2 is independent of η_5 , η_7 and η_8 . Furthermore, we apply (2.50), (2.53) and (2.55) to f_2 and obtain

$$0 = E_{\bar{\alpha}_3}(f_2) = -3\vartheta \partial_{\eta_3}(f_2), \quad 0 = E_{\bar{\alpha}_2 + \bar{\alpha}_3}(f_2) = 3\vartheta \partial_{\eta_4}(f_2), \quad (3.59)$$

$$0 = E_{\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3}(f_2) = -3\vartheta \partial_{\eta_6}(f_2). \quad (3.60)$$

Therefore, f_2 is a rational function in $x_1, x_2, \zeta_1, \zeta_2, \eta_1, \eta_2$. By (2.51),

$$0 = E_{\bar{\alpha}_4}(f_2) = -x_1 \partial_{x_2}(f_2) - \zeta_1 \partial_{\zeta_2}(f_2). \quad (3.61)$$

Again the characteristic method tell us that f_2 can be written as a rational function f_3 in $x_1, \zeta_1, \vartheta, \eta_1, \eta_2$. Since $f_2 = f$ is a polynomial in $\{x_r \mid r \in \overline{1, 26}\}$, Expressions (3.29), (3.34), (3.42) and (3.43) imply that f_2 must be a polynomial in $x_1, \zeta_1, \vartheta, \eta_1, \eta_2$. The other statements follow directly. \square

Calculating the weights of the singular vectors in the above theorem, we have:

Corollary 3.2. *The space of polynomial \mathcal{G}^{F_4} -invariants over its basic module is an subalgebra of \mathcal{A} generated by η_1 and η_2 .*

Let $L(m_1, m_2, m_3, m_4, m_5)$ be the \mathcal{G}^{F_4} -submodule generated by $x_1^{m_1} \zeta_1^{m_2} \vartheta^{m_3} \eta_1^{m_4} \eta_2^{m_5}$. Then $L(m_1, m_2, m_3, m_4, m_5)$ is a finite-dimensional irreducible \mathcal{G}^{F_4} -submodule with the highest weight $m_3 \lambda_3 + (m_1 + m_2) \lambda_4$. Let \mathcal{A}_k be the subspace of polynomials in \mathcal{A} with degree k . Then \mathcal{A}_k is a finite-dimensional \mathcal{G}^{F_4} -module. By the Weyl's theorem of complete reducibility,

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k = \bigoplus_{m_1, m_2, m_3, m_4, m_5=0}^{\infty} L(m_1, m_2, m_3, m_4, m_5). \quad (3.62)$$

Denote by $d(k, l)$ the dimension of the highest weight irreducible module with the weight $k \lambda_3 + l \lambda_4$. The above equation imply the following combinatorial identity:

$$\frac{1}{(1-t)^{26}} = \frac{1}{(1-t^2)(1-t^3)} \sum_{k_1, k_2, k_3=0}^{\infty} d(k_1, k_2 + k_3) t^{3k_1 + 2k_2 + k_3}. \quad (3.63)$$

Multiplying $(1-t)^2$ to the above equation, we obtain a new combinatorial identity about twenty-four:

$$\frac{1}{(1-t)^{24}} = \frac{1}{(1+t)(1+t+t^2)} \sum_{k_1, k_2, k_3=0}^{\infty} d(k_1, k_2 + k_3) t^{3k_1 + 2k_2 + k_3}. \quad (3.64)$$

Equivalently, we have:

Corollary 3.3. *The dimensions $d(p, l)$ of the irreducible module with the weights $k \lambda_3 + l \lambda_4$ are linearly correlated by the following identity:*

$$(1+t)(1+t+t^2) = (1-t)^{24} \sum_{k_1, k_2, k_3=0}^{\infty} d(k_1, k_2 + k_3) t^{3k_1 + 2k_2 + k_3}. \quad (3.65)$$

In the construction of the root system Φ_{F_4} from Euclidean space (e.g.,cf. [Hu]),

$$(\lambda_1, \bar{\alpha}_1) = (\lambda_2, \bar{\alpha}_2) = 1, \quad (\lambda_1, \bar{\alpha}_3) = (\lambda_2, \bar{\alpha}_4) = 1/2. \quad (3.66)$$

Recall $\delta = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. By the dimension formula of finite-dimensional irreducible modules of simple Lie algebras (e.g.,cf. [Hu]),

$$\begin{aligned} d(k, l) &= \frac{\prod_{\bar{\alpha} \in \Phi_{F_4}^+} (\lambda + \delta, \bar{\alpha})}{\prod_{\bar{\alpha} \in \Phi_{F_4}^+} (\delta, \bar{\alpha})} = \frac{(l+1)(k+3)(k+l+4)}{39504568320000} (2k+l+7)(3k+l+10) \\ &\quad \times (3k+2l+11) \left(\prod_{r=1}^5 (k+r) \right) \left(\prod_{s=2}^6 (k+l+s) \right) \left(\prod_{q=5}^9 (2k+l+q) \right). \end{aligned} \quad (3.67)$$

Recall the quadratic invariants η_1 in (3.4). Dually we have \mathcal{G}^{F_4} invariant complex Laplace operator

$$\Delta_{F_4} = 3 \sum_{r=1}^{12} \partial_{x_r} \partial_{x_{\bar{r}}} - \partial_{x_{13}}^2 - \partial_{x_{13}} \partial_{x_{14}} - \partial_{x_{14}}^2. \quad (3.68)$$

Now the subspace of complex homogeneous harmonic polynomials with degree k is

$$\mathcal{H}_k^{F_4} = \{f \in \mathcal{A}_k \mid \Delta_{F_4}(f) = 0\}. \quad (3.69)$$

Then $\mathcal{H}_1^{F_4} = V$. Assume $k \geq 2$. Suppose that k_1, k_2, m_1, m_2 are nonnegative integers such that

$$k_1 + 3k_2 = m_1 + 3m_2 + 2 = k. \quad (3.70)$$

If $\Delta_{F_4}(x_1^{k_1}\vartheta^{k_2}) \neq 0$, then it is a singular vector of degree $k - 2$ with weight $k_2\lambda_3 + k_1\lambda_4$. By Theorem 3.1, \mathcal{A}_{k-2} does not contain a singular vector of such weight. A contradiction. Thus $\Delta_{F_4}(x_1^{k_1}\vartheta^{k_2}) = 0$. By the same reason, $\Delta_{F_4}(x_1^{m_1}\zeta_1\vartheta^{m_2}) = 0$. Thus the irreducible submodules

$$L(k_1, 0, k_2, 0, 0), L(m_1, 1, m_2, 0, 0, 0) \subset \mathcal{H}_k^{F_4}. \quad (3.71)$$

This gives the following corollary:

Corollary 3.4. *The number of irreducible submodules contained in the subspace $\mathcal{H}_k^{F_4}$ of complex homogeneous harmonic polynomials with degree $k \geq 2$ is $\geq \lfloor k/3 \rfloor + \lfloor (k-2)/3 \rfloor + 2$.*

We remark that the above conclusion implies a similar conclusion on real harmonic polynomials for the real compact simple Lie algebras of type F_4 .

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